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Extended cohomological field theories and noncommutative Frobenius manifolds

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Abstract

We construct some extension of cohomological field theories (*stable field theory*). The stable field theory is a system of homomorphisms to some vector spaces generated by spheres and disks with punctures. It is described by a formal tensor series, satisfying to some system of “differential equations”. In points of convergence the tensor series generate special noncommutative analogues of Frobenius algebras, describing ‘open-closed’ topological field theories.

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1. Introduction

Cohomological field theories were proposed by Kontsevich and Manin [7] for description of Gromov–Witten classes. They proved that cohomological field theory is equivalent to formal Frobenius manifold. Formal Frobenius manifold is defined by a formal series F , satisfying the associative equations [5,14]. In points of convergence the series F defines Frobenius algebras. The set of these points forms a Frobenius manifold as regards to Dubrovin [6].

Cohomological field theory is a system of special homomorphisms to spaces of cohomology of Deligne–Mumford compactifications for moduli spaces of complex rational curves (Riemann spheres) with punctures. In this paper it is constructed some extension of cohomological field theories. This extension (*stable field theory*) is a system of homomorphisms

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to some vector spaces generated by disks with punctures. I conjecture that they describe relative Gromov–Witten classes.

A stable field theory is equivalent to some analogue of a formal Frobenius manifold. This analogue is defined by formal tensor series (*structure series*), satisfying some system of “differential equations” (including the associativity equation). In points of convergence the structure series define *extended Frobenius algebras*. They are special noncommutative analogue of Frobenius algebras. Extended Frobenius algebras describe ‘open-closed’ topological field theories [8] of genus 0 in the same way as Frobenius algebras describe Atiyah–Witten 2D topological field theories [2,13]. Thus, structure series are noncommutable analogues of formal Frobenius manifolds.

In Sections 2 and 3 it is proposed a general axiomatic of topological field theories over functors. This class involves 2D Atiyah–Witten [2,13], ‘open-closed’ [8], Klein topological field theories [3] and cohomological field theories [7,9].

In Section 4 it is constructed and investigated a *stabilising functor* on a category of spheres with punctures, disks with punctures and its disconnected unions. For spheres this construction is linked to modular graphs [9] and describes a Cohomological functor for complex rational curves. The stable field theories are defined as topological field theories over stabilising functors.

In Sections 5 and 6 we prove that stable field theories are in one-to-one correspondence to *systems of disk correlation functions*. Its “generating functions” are the structure series.

In Sections 7 and 8 it is demonstrated that structure series generate extended Frobenius algebras in their convergence points and it is constructed some examples.

2. 2D categories

2.1 In this paper a *surface* denotes a compact surface with or without boundary. Its connected boundary component is called a *boundary contour* of the surface. Any orientable connected surface is homeomorphic to a sphere with g handles and s holes. Such surface is called a *surface of type* $(g, s, 1)$. Any nonorientable connected surface is homeomorphic to either a projective plane with a handles and s holes or a Klein bottle with a handles and s holes. Such surface is called a *surface of type* $(g, s, 0)$, where $g = a + (1/2)$ in the first case and $g = a + 1$ in the second case.

A surface with a finite number of marked points is called a *stratified surface*. Marked points are also called *special points* of the stratified surface. Two stratified surfaces are called *isomorphic*, if there exists a homeomorphism of the surfaces, generating a bijection between their special points.

By $|S|$ we denote the cardinality of a finite set S . Let Ω be a surface of the type (g, s, ϵ) and $\omega_1, \dots, \omega_s$ be its boundary contours. Let $S \subset \Omega$ be a finite set of marked points and $m = |S \cap (\Omega \setminus \partial\Omega)|$, $m_i = |S \cap \omega_i|$. Then the collection $G = (g, \epsilon, m, m_1, \dots, m_s)$ is called a *type* of connected stratified surface (Ω, S) . A connected stratified surface of type G is called *trivial*, if $\mu = 2g + m + s + (1/2) \sum m_i - 2 \leq 0$. It is easy to prove the following statement.

Lemma 2.1. *Any trivial stratified surface is isomorphic to a stratified surface from the list:*

- (1) a sphere S^2 without special points ($\mu = -2$);
- (2) a projective plane $\mathbb{R}P^2$ without special points ($\mu = -1$);
- (3) a disk D^2 without special points ($\mu = -1$);
- (4) a sphere (S^2, p) with a single interior special point p ($\mu = -1$);
- (5) a disk (D^2, q) with a single special boundary point and without special interior points ($\mu = -1/2$);
- (6) a sphere (S^2, p_1, p_2) with two interior special points ($\mu = 0$);
- (7) a projective plane ($\mathbb{R}P^2, p$) with a single interior special point ($\mu = 0$);
- (8) a torus T^2 without special points ($\mu = 0$);
- (9) a Klein bottle Kl without special points ($\mu = 0$);
- (10) a disk (D^2, p) with a single interior special point and without boundary special points ($\mu = 0$);
- (11) a disk (D^2, q_1, q_2) with two boundary special points and without interior special points ($\mu = 0$);
- (12) a Möbius band Mb without special points ($\mu = 0$);
- (13) a cylinder Cyl without special points ($\mu = 0$).

Let Ω be a stratified surface. A generic not self-intersecting curve $\gamma \subset \Omega$ is called a *cut*. Generic means that γ has no special points and either it is a (closed) contour without boundary points of Ω or it is a segment, whose ends belong to the boundary of Ω and all interior points are interior points of the surface. The cuts form nine topological classes described in [3]. A set of pairwise nonintersecting cuts is called a *cut system*.

Let Γ be a cut system of a stratified surface Ω . Consider compactification $\tilde{\Omega}$ of $\Omega \setminus \Gamma$ by pairs (x, c) , where $x \in \gamma \subset \Gamma$ and c is a coorientation of the cut γ in a neighbourhood of x . Denote by Ω/Γ a surface obtained by contracting each connected component C_i of $\tilde{\Omega} \setminus (\Omega \setminus \Gamma)$ into a point c_i . We assume that $\Omega_{\#} = \Omega/\Gamma$ is a stratified surface. Its special points are the special points of Ω and the points c_i .

2.2 In this section, following [3], we define a tensor category \mathcal{C} of stratified surfaces with a set \mathcal{O} of local orientations of special points. A *set of local orientations* means that for any special point $r \in Q$ of stratified surface Ω we fix an orientation or of its small neighbourhood.

A set \mathcal{O} of local orientations is said *admissible*, iff either Ω is orientable surface and all local orientations are induced by an orientation of Ω or Ω is nonorientable surface and all local orientations at all special points from any boundary contour ω_i are compatible with one of the orientations of ω_i . Moreover, we consider that any boundary contour contain at least one special point.

Lemma 2.2. *Let (Ω', \mathcal{O}') and $(\Omega'', \mathcal{O}'')$ be two pairs, consisting of stratified surfaces and admissible sets of local orientations at special points. If stratified surfaces Ω' and Ω'' are isomorphic then there exists an isomorphism $\phi : \Omega' \rightarrow \Omega''$ such that $\phi(\mathcal{O}') = \mathcal{O}''$.*

The proof follows from standard properties of surfaces.

Pairs (Ω, \mathcal{O}) of stratified surfaces Ω with sets of local orientations \mathcal{O} at their special points are *objects of the basic category C*. *Morphisms* are any combinations of the morphisms of types (1)–(4).

- (1) *Isomorphism* $\phi : (\Omega, \mathcal{O}) \rightarrow (\Omega', \mathcal{O}')$. By a definition, ϕ is an isomorphism $\phi : \Omega \rightarrow \Omega'$ of stratified surfaces compatible with local orientations at special points.
- (2) *Change of local orientations* $\psi : (\Omega, \mathcal{O}) \rightarrow (\Omega, \mathcal{O}')$. Thus, there is one such morphism for any pair $(\mathcal{O}, \mathcal{O}')$ of sets of local orientations on a stratified surface Ω .
- (3) *Cutting* $\eta : (\Omega, \mathcal{O}) \rightarrow (\Omega_\#, \mathcal{O}_\#)$. The morphism η depends on a cut system Γ endowed with orientations of all cuts $\gamma \in \Gamma$. $\Omega_\#$ is defined as contracted cut surface Ω/Γ . Stratified surface $\Omega_\#$ inherits special points of Ω and local orientations at any of them. The orientations of cuts induce the local orientations at other special points.

Define a ‘*tensor product*’ $\theta : (\Omega', \mathcal{O}') \times (\Omega'', \mathcal{O}'') \rightarrow (\Omega, \mathcal{O})$ of two pairs (Ω', \mathcal{O}') and $(\Omega'', \mathcal{O}'')$ as their disjoint union $(\Omega, \mathcal{O}) = (\Omega' \sqcup \Omega'', \mathcal{O}' \sqcup \mathcal{O}'')$.

Subcategories of the basis category \mathcal{C} are called *2D categories*. The basic category \mathcal{C} has subcategories $\mathcal{C}_{g,s,\epsilon}$, where g is either an integer or half-integer non-negative number or ∞ , s is an integer non-negative number or ∞ and $\epsilon = 0, 1$. The objects of $\mathcal{C}_{g,0,0}$ are all pairs (Ω, \mathcal{O}) , where Ω is a stratified surface of type (\tilde{g}, ϵ, m) and $\tilde{g} \leq g$. For $s > 0$ objects of $\mathcal{C}_{g,s,0}$ are all pairs (Ω, \mathcal{O}) , where Ω is a stratified surface of type $(\tilde{g}, \epsilon, m, m_1, \dots, m_{\tilde{s}})$, $\tilde{g} \leq g$, and $\tilde{s} \leq 2(g - \tilde{g}) + s$. Thus, $\mathcal{C} = \mathcal{C}_{\infty,\infty,0}$. The category $\mathcal{C}_{g,s,1}$ is a subcategory of $\mathcal{C}_{g,s,0}$. Its objects are all objects (Ω, \mathcal{O}) of $\mathcal{C}_{g,s,0}$ such that Ω is an orientable surface. Denote by $\mathcal{C}_{g,s,1,0}$ subcategory of $\mathcal{C}_{g,s,1}$, consisting of (Ω, \mathcal{O}) , where \mathcal{O} is generated by some global orientation of Ω . This global orientation is marked by the same symbol \mathcal{O} .

2.3 Below we define a *structure functor* $(\Omega, \mathcal{O}) \rightarrow V(\Omega, \mathcal{O})$ from the basic category of surfaces to the category of vector spaces [3].

Let $\{X_m | m \in M\}$ be a finite set of $n = |M|$ vector spaces X_m . The action of the symmetric group S_n on the set $\{1, \dots, n\}$ induces its action on the linear space $(\oplus_{\sigma: \{1, \dots, n\} \leftrightarrow M} X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)})$, an element $s \in S_n$ brings a summand $X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)}$ to the summand $X_{\sigma(s(1))} \otimes \dots \otimes X_{\sigma(s(n))}$. Denote by $\otimes_{m \in M} X_m$ the subspace of all invariants of this action.

The vector space $\otimes_{m \in M} X_m$ is canonically isomorphic to a tensor product of all X_m in any fixed order, the isomorphism is a projection of the vector space $\otimes_{m \in M} X_m$ to the summand that is equal to the tensor product of X_m in a given order. Assume that all X_m are equal to a fixed vector space X . Then any bijection $M \leftrightarrow M'$ of sets induces the isomorphism $\otimes_{m \in M} X_m \leftrightarrow \otimes_{m' \in M'} X_{m'}$.

Let A and B be finite dimensional vector spaces over a field \mathbb{K} endowed with involutive linear transformations $A \rightarrow A$ and $B \rightarrow B$, which we denote by $x \mapsto x^*$ ($x \in A$) and $y \mapsto y^*$ ($y \in B$), resp.

Let (Ω, \mathcal{O}) be a pair, consisting of a stratified surface Ω and a set \mathcal{O} of local orientations at its special points. Denote by Ω_a the set of all interior special points and by Ω_b the set of all boundary special points. Put also $\Omega_0 = \Omega_a \sqcup \Omega_b$. Assign a copy A_p of a vector space A to any point $p \in \Omega_a$ and a copy B_q of a vector space B to any point $q \in \Omega_b$. Put $V(\Omega, \mathcal{O}) = V_\Omega = (\otimes_{p \in \Omega_a} A_p) \otimes (\otimes_{q \in \Omega_b} B_q)$.

The group $\Sigma(\Omega)$ of transpositions of Ω_b acts natural on V_Ω .

For any morphism of pairs $(\Omega, \mathcal{O}) \rightarrow (\Omega', \mathcal{O}')$ define a morphism of vector spaces $V_\Omega \rightarrow V_{\Omega'}$ as follows:

- (1) An isomorphism $\phi : (\Omega, \mathcal{O}) \rightarrow (\Omega', \mathcal{O}')$ induces the isomorphism $\phi_* : V_\Omega \rightarrow V_{\Omega'}$ because ϕ generates the bijections $\Omega_a \leftrightarrow \Omega'_a$ and $\Omega_b \leftrightarrow \Omega'_b$ of sets of special points.

- (2) For a change of local orientations $\psi : (\Omega, \mathcal{O}) \rightarrow (\Omega, \mathcal{O}')$ define a linear map $\psi_* : V_\Omega \rightarrow V_\Omega$ as $(\otimes_{r \in \Omega_0} \psi_r)$, where for any $r \in \Omega_0$

$$\psi_r(x) = \begin{cases} x & \text{if } o_r = o'_r, \\ x^* & \text{if } o_r = -o'_r. \end{cases}$$

- (3) In order to define a morphism $\eta_* : V_\Omega \rightarrow V_{\Omega_\#}$ for any cutting morphism $\eta : (\Omega, \mathcal{O}) \rightarrow (\Omega_\#, \mathcal{O}_\#)$ we need to fix elements $\hat{K}_{A,*} \in A \otimes A$, and $\hat{K}_{B,*} \in B \otimes B$ and $U \in A$. (The notation will be clear from the sequel.)

Evidently, it is sufficient to define η_* for an arbitrary oriented cut $\gamma \subset \Omega$. In this case we have a canonical isomorphism $V_{\Omega_\#} = V_\Omega \otimes X$, where

$$X = \begin{cases} A \otimes A & \text{if } \gamma \text{ is a coorientable contour,} \\ B \otimes B & \text{if } \gamma \text{ is a segment,} \\ A & \text{if } \gamma \text{ is a noncoorientable cut.} \end{cases}$$

For $x \in V_\Omega$ put $\eta_*(x) = x \otimes z$, where z is either $\hat{K}_{A,*}$, or $\hat{K}_{B,*}$, or U resp.

Finally, for a ‘tensor product’ $\theta : (\Omega', \mathcal{O}') \times (\Omega'', \mathcal{O}'') \rightarrow (\Omega' \sqcup \Omega'', \mathcal{O}' \sqcup \mathcal{O}'')$ there is evident canonical linear map $\theta_* : V_{\Omega'} \otimes V_{\Omega''} \rightarrow V_{\Omega' \sqcup \Omega''}$.

3. Topological field theory over a functor

3.1 Let \mathcal{R} be a category of triples (W, ρ, Σ) , where W is a vector spaces over a field \mathbb{K} , Σ is a group, and $\rho : \Sigma \rightarrow \text{Aut}(W)$ is a homomorphism. Morphism $(W, \rho, \Sigma) \rightarrow (W', \rho', \Sigma')$ is a pair of isomorphisms $(\vartheta_W : W \rightarrow W', \vartheta_\Sigma : \Sigma \rightarrow \Sigma')$ such that $\vartheta_W \rho = \rho' \vartheta_\Sigma$.

Let \mathcal{T} be a functor from a 2D category to \mathcal{R} such that $\mathcal{T}(\Omega, \mathcal{O}) = (W(\Omega, \mathcal{O}), \rho(\Omega, \mathcal{O}), \Sigma(\Omega, \mathcal{O}))$, where $\Sigma(\Omega, \mathcal{O}) = \Sigma(\Omega)$ is the group of transpositions of Ω_b .

We consider that $\mathcal{T}(\Omega, \mathcal{O}) = (\mathbb{K}, \rho, \Sigma)$, where $\rho(\Sigma)$ is the identical map, if Ω is a trivial stratified surface, and $\mathcal{T}(\pi)$ is the identical morphism, if π is a morphism of trivial stratified surfaces.

A topological field theory over \mathcal{T} is a set

$$\mathcal{F} = \{A, x \mapsto x^*, B, y \mapsto y^*, \mathcal{T}, \Phi_{(\Omega, \mathcal{O})}\},$$

where $(A, x \mapsto x^*)$ and $(B, y \mapsto y^*)$ are finite dimensional vector spaces over \mathbb{K} endowed with involute linear transformation and $\{\Phi_{(\Omega, \mathcal{O})}\}$ is a family of linear operators $\Phi_{(\Omega, \mathcal{O})} : V_\Omega \rightarrow W(\Omega, \mathcal{O})$, where V_Ω is the image of (Ω, \mathcal{O}) by the structural functor for $\{A, x \mapsto x^*, B, y \mapsto y^*\}$ and $\mathcal{T}(\Omega, \mathcal{O}) = (W(\Omega, \mathcal{O}), \rho(\Omega, \mathcal{O}), \Sigma(\Omega, \mathcal{O}))$.

The set \mathcal{F} is called a topological field theory, if the following axioms are satisfied:

- (0) Algebraic invariance

For any $\sigma \in \Sigma(\Omega, \mathcal{O})$ it is required that

$$\Phi_{(\Omega, \mathcal{O})}(\sigma(x)) = \rho(\Omega, \mathcal{O})(\sigma)(\Phi_{(\Omega, \mathcal{O})}(x)).$$

(1) *Topological invariance*

For any isomorphism of pairs $\phi : (\Omega, \mathcal{O}) \rightarrow (\Omega', \mathcal{O}')$ it is required that

$$\Phi_{(\Omega', \mathcal{O}')}(\phi_*(x)) = \mathcal{T}(\phi)\Phi_{(\Omega, \mathcal{O})}(x).$$

(2) *Invariance of a change of local orientations*

For any change of local orientations $\psi : (\Omega, \mathcal{O}) \rightarrow (\Omega, \mathcal{O}')$ it is required that

$$\Phi_{(\Omega, \mathcal{O}')}(\psi_*(x)) = \mathcal{T}(\psi)\Phi_{(\Omega, \mathcal{O})}(x).$$

(3) *Nondegeneracy*

Define first a bilinear form $(x, x')_A$ on the vector space A . Namely, let (Ω, \mathcal{O}) be a pair, where Ω is a sphere with two interior special points p, p' and the set $\mathcal{O} = \{o_p, o_{p'}\}$ is such that local orientations $o_p, o_{p'}$ induce the same global orientations of the sphere. Put $(x, x')_A = \Phi_{(\Omega, \mathcal{O})}(x_p \otimes x'_{p'})$, where x_p and $x'_{p'}$ are images of $x \in A$ and $x' \in A$ in A_p and $A_{p'}$ resp. The correctness of this definition follows from [Axioms 5.1 and 5.2](#). Evidently, $(x, x')_A$ is a symmetric bilinear form.

Similarly, define a bilinear form $(y, y')_B$ on the vector space B , using a disc with two boundary special points q, q' instead of a sphere with two interior special points p, p' . As in the previous case, local orientations $o_q, o_{q'}$ must induce the same global orientations of the disc. Evidently, $(y, y')_B$ is a symmetric bilinear form.

It is required that forms $(x, x')_A$ and $(y, y')_B$ are nondegenerate.

(4) *Cut invariance*

[Axioms 5.1–5.3](#) allow us to choose elements $\hat{K}_{A,*} \in A \times A, \hat{K}_{B,*} \in B \times B$ and $U \in A$. Indeed, any nondegenerate bilinear form on the vector space X canonically defines [[3, Section 2](#)] the tensor Casimir element $\hat{K}_X \in X \otimes X$. Taking forms $(x, x')_{A,*} = (x, x^{*})_A$ and $(y, y')_{B,*} = (y, y^{*})_B$, we obtain elements $\hat{K}_{A,*}$ and $\hat{K}_{B,*}$.

A linear form $\Phi_{(\Omega, \mathcal{O})}$ for a projective plane Ω with one interior special point is an element of the vector space dual to A . We denote by U the image of this element in A under the isomorphism induced by nondegenerate bilinear form $(x, x')_A$.

We shall use just these elements for morphisms η_* of type (3).

For any cut system Γ endowed with orientations of all cuts it is required that

$$\Phi_{(\Omega_{\#}, \mathcal{O}_{\#})}(\eta_*(x)) = \mathcal{T}(\eta)\Phi_{(\Omega, \mathcal{O})}(x).$$

(5) *Multiplicativity*

For the product $\theta : (\Omega', \mathcal{O}') \times (\Omega'', \mathcal{O}'') \rightarrow (\Omega' \sqcup \Omega'', \mathcal{O}' \sqcup \mathcal{O}'')$ of any two pairs (Ω', \mathcal{O}') and $(\Omega'', \mathcal{O}'')$ it is required that

$$\Phi_{(\Omega_1 \sqcup \Omega_2, \mathcal{O}_1 \sqcup \mathcal{O}_2)}(\theta_*(x_1 \otimes x_2)) = \Phi_{(\Omega_1, \mathcal{O}_1)}(x_1) \otimes \Phi_{(\Omega_2, \mathcal{O}_2)}(x_2).$$

3.2 Let us consider now some example of topological field theory on 2D categories. In this section we consider only functors $\mathcal{T}(\Omega, \mathcal{O}) = (W(\Omega, \mathcal{O}), \rho(\Omega, \mathcal{O}), \Sigma(\Omega, \mathcal{O}))$, such that $\rho(\Omega, \mathcal{O})(\sigma)$ is the identical map for all $\sigma \in \Sigma(\Omega, \mathcal{O})$.

(1) *Topological field theory on $\mathcal{C}_{g,s,\epsilon}$ (over trivial functor)*

Consider the functor, corresponding the field \mathbb{K} to all objects (Ω, \mathcal{O}) and corresponding identical map to any morphism:

- (a) This gives an Atiyah–Witten 2D topological field theory [2,13] for the category $\mathcal{C}_{\infty,0,1}$ and involution $x^* = x$.
- (b) The category $\mathcal{C}_{\infty,\infty,1}$ and the involutions $x^* = x, y^* = y$ gives ‘an open-closed topological field theory in the sense of Lazaroiu [8].
- (c) The category $\mathcal{C}_{\infty,\infty,0}$ gives a Klein topological field theory in the sense of [3] (without units).

(2) Cohomological field theories on $\mathcal{C}_{g,s,\epsilon}$

Recall that a Klein surface of type (g, s, ϵ) is a surface of type (g, s, ϵ) , endowed with a dianalytic structure, i.e., an atlas with holomorphic and antiholomorphic transitions functions [1]. It is equivalent to a real algebraic curve (for an information about real algebraic curves, see [11]). A Klein surface of type $G = (g, \epsilon, m, m_1, \dots, m_s)$ is called a Klein surface with special points of type G . The moduli space of Klein surfaces is constructed in [10]. Let $H^*(\bar{M}_G, \mathbb{K})$ be a cohomological algebra of Deligne–Mumford compactification of the space of Klein surfaces of type G . An identification of special points gives some embeddings $\bar{M}_{G_1} \times \bar{M}_{G_2} \rightarrow \bar{M}_G$ and $\bar{M}_{G_1} \rightarrow \bar{M}_G$ by analogy with [9].

Cohomological field theory in our conception is defined as the topological field theory over the functor \mathcal{T} that associate the algebra $H^*(\bar{M}_G, \mathbb{K})$ with each object (Ω, \mathcal{O}) , where G is the type of Ω , and associate the homomorphisms generated by the embeddings $\bar{M}_{G_1} \times \bar{M}_{G_2} \rightarrow \bar{M}_G$ and $\bar{M}_{G_1} \rightarrow \bar{M}_G$ with the cutting morphisms. Here $x^* = x, y^* = y$ and the other morphisms are the same as that for the trivial functor.

This definition takes the (complete) cohomological field theory as regards to [9] for subcategories $\mathcal{C}_{0,0,1}$ ($\mathcal{C}_{\infty,0,1}$).

3.3 According to [9], the cohomological field theory over $\mathcal{C}_{0,0,1}$ generate some deformations of Atiyah–Witten 2D topological field theory in genus 0. These deformations are described by formal Frobenius manifolds, i.e., formal solutions of WDVV equations [5,14].

Our goal is the construction of a functor on $\mathcal{C}_{0,1,1}^0$ such that topological field theory over this functor generate deformations of open-closed topological field theories in genus 0. We shall prove that these deformations are described by formal solutions of some noncommutative analogues of WDVV equations.

4. Stable field theories

4.1 Let $\Omega = (\Omega, \mathcal{O})$ be an oriented stratified surface and \mathcal{O} be generated by some orientation. (Here and later we omit the mark of orientation \mathcal{O} , if it is constant in the construction). The orientation of Ω gives a natural sense to inequalities $b_1 < b_2 < b_3$, for points from a connected component of the boundary of Ω . For $C \subset \Omega_b$ denote by $\Sigma^\circ(C)$ the group of transpositions $\sigma \in \Sigma(\Omega)$, such that $\sigma(b_1) < \sigma(b_2) < \sigma(b_3)$ if $b_1 < b_2 < b_3$ and $b_1, b_2, b_3 \in C$.

A pair (Γ, σ) , where $\Gamma \subset \Omega$ is a cut system and $\sigma \in \Sigma(\Omega)$ is called a tiling, if all connected components of Ω/Γ are nontrivial. A tiling (Γ, σ) , where $\Gamma = \emptyset$ is also assumed and it is called empty tiling.

Two tiling (Γ', σ') and (Γ'', σ'') are called isomorphic, if there exists a homeomorphism $\psi : \Omega \rightarrow \Omega$, preserving the orientation, moving σ' to σ'' and such that $\psi(\Gamma') = \Gamma''$. An

isomorphic class of a tiling is called a *diagram*. Let $[\Gamma, \sigma]$ be the diagram corresponding to a tiling (Γ, σ) .

Denote by $\mathbb{K}(\Omega)$ the vector space over \mathbb{K} generated by all diagrams of Ω .

Consider a tiling (Γ, σ) , corresponding to a diagram $T \in \mathbb{K}(\Omega)$. A connected component of $\Omega \setminus \Gamma$ is called a *vertex* of the tiling. Denote by $V(T)$ the set of all vertices of T .

Let Ω_v^T be the connected component of Ω/Γ , corresponding to a vertex v . Let A_v^T and B_v^T be the set of interior and boundary special points of Ω_v^T . We can associate $\sigma^* \in \Sigma(\Omega)$ to any $\sigma_v^T \in \Sigma(\Omega_v^T)$, considering that $\sigma^*(b_1) < \sigma^*(b_2) < \sigma^*(b_3)$ if $b_1, b_2, b_3 \notin \Omega_v^T$ belong to the same connected component of $\Omega \setminus \Gamma$ and $b_1 < b_2 < b_3$. Moreover, we can consider any cut system on $\Gamma_v^T \in \Omega_v^T$ as a cut system on Ω . Thus a diagram $S = [\Gamma_v^T, \sigma_v^T] \in \mathbb{K}(\Omega_v^T)$ generate a diagram $(T, v, S) = [\Gamma \cup \Gamma_v^T, \sigma^* \sigma] \in \mathbb{K}(\Omega)$.

4.2 A tiling (Γ, σ) is called *simple*, if Γ consist of one cut. A simple tiling is called *A-tiling*, if Γ is a closed contour homotopic to zero, *B-tiling*, if Γ is a segment, dividing the surface, *C-tiling*, if Γ is a closed contour nonhomotopic to zero, and *D-tiling*, if Γ is a segment, nondividing the surface. Diagrams corresponding to $A(B, C, D)$, *empty* tillings are called $A(B, C, D)$, *empty diagram*.

Let C_1 and C_2 be subsets of a surface Ω . By $r_A(C_1|C_2)$ (respectively, $r_B(C_1|C_2)$) denote the set of A - (respectively, B -) diagrams corresponding to (Γ, σ) , where Γ divide C_1 and C_2 .

For a sphere Ω and $a_1, a_2, a_3, a_4 \in \Omega_a$ put $r_\Omega(a_1, a_2|a_3, a_4) = r_A(a_1, a_2|a_3, a_4)$.

Let Ω be a disk, $b_1, b_2, b_3, b_4 \in \Omega_b$ and $b_1 < b_2 < b_3 < b_4$. Denote by $r_\Omega(b_1, b_2|b_3, b_4)$ the set of all diagrams belonging to $r_B(\sigma(b_1) \cup \sigma(b_2)|\sigma(b_3), \cup \sigma(b_4))$ and corresponding to (Γ, σ) , where $\sigma \in \Sigma^\circ(\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3) \cup \sigma(b_4)) \cap \Sigma(\Omega_b - (\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3) \cup \sigma(b_4)))$.

Let Ω be a disk $a \in \Omega_a, b_1, b_2 \in \Sigma(\Omega)$. Denote by $r_\Omega(b_1, b_2|a)$ the set of all diagrams belonging to $r_B(\sigma(b_1) \cup \sigma(b_2)|a)$ and corresponding to (Γ, σ) , where $\sigma \in \Sigma(\Omega_b - (\sigma(b_1) \cup \sigma(b_2)))$ and any ends b of Γ satisfy $\sigma(b_1) < b < \sigma(b_2)$.

Let Ω be a disk $a_1, a_2 \in \Omega_a, b \in \Omega_b$. Put $r_\Omega(a_1, a_2|b) = r_A(a_1, a_2|b)$. Denote by $r_\Omega(a_1, b|a_2)r_\Omega(a_2, b|a_1)$ the diagrams corresponding to $(\Gamma_1 \cup \Gamma_2, \sigma)$ where (Γ_1, σ) from $r_B(a_1 \cup \sigma(b)|a_2)$, (Γ_2, σ') from $r_B(a_2 \cup \sigma(b)|a_1)$, and $\sigma \in \Sigma(\Omega_b - b)$.

Let Ω be a cylinder. Denote by $r_\Omega^\sigma(C)$ and $r_\Omega^\sigma(D)$ the set of all C - and D -diagrams respectively.

Denote by $H^*(\Omega)$ the set of linear functions $l : \mathbb{K}(\Omega) \rightarrow \mathbb{K}$, that is equal to 0 on all elements in the forms:

- (1) $[\phi(\Gamma), \sigma(\phi)\sigma] - [\Gamma, \sigma]$
for any tiling (Γ, σ) of Ω , any isomorphism $\phi : (\Omega, \mathcal{O}) \rightarrow (\Omega, \mathcal{O})$ and the transposition $\sigma(\phi) = \phi|_{\Omega_b}$
- (2) $\sum_{S \in r_{\Omega_v^T}(a_1, a_2|a_3, a_4)} (T, v, S) - \sum_{S \in r_{\Omega_v^T}(a_2, a_3|a_4, a_1)} (T, v, S)$
for any diagram $T \in \mathbb{K}(\Omega)$, any $v \in V(T)$ such that Ω_v^T , is a sphere, and any pairwise different points $a_1, a_2, a_3, a_4 \in (\Omega_v^T)_a$.
- (3) $\sum_{S \in r_{\Omega_v^T}(b_1, b_2|b_3, b_4)} (T, v, S) - \sum_{S \in r_{\Omega_v^T}(b_2, b_3|b_4, b_1)} (T, v, S)$

for any diagram $T \in \mathbb{K}(\Omega)$, any $v \in V(T)$ such that Ω_v^T is a disk and any $b_1, b_2, b_3, b_4 \in (\Omega_v^T)_b$ such that $b_1 < b_2 < b_3 < b_4$.

$$(4) \quad \sum_{S \in r_{\Omega_v^T}(b_1, b_2 | a)} (T, v, S) - \sum_{S \in r_{\Omega_v^T}(b_2, b_1 | a)} (T, v, S)$$

for any diagram $T \in \mathbb{K}(\Omega)$, any $v \in V(T)$ such that Ω_v^T is a disk any $a \in (\Omega_v^T)_a$, and any different points $b_1, b_2 \in (\Omega_v^T)_b$.

$$(5) \quad \sum_{S \in r_{\Omega_v^T}(a_1, a_2 | b)} (T, v, S) - \sum_{S \in r_{\Omega_v^T}(a_1, b | a_2) r_{\Omega_v^T}(a_2, b | a_1)} (T, v, S)$$

for any diagram $T \in \mathbb{K}(\Omega)$, any $v \in V(T)$ such that Ω_v^T is a disk with $n + 2$ interior and $m + 1$ boundary marked points, $a \in (\Omega_v^T)_a$, and distinct $b_1, b_2 \in (\Omega_v^T)_b$.

$$(6) \quad \sum_{S \in r_{\Omega_v^T}(C)} (T, v, S) - \sum_{S \in r_{\Omega_v^T}(D)} (T, v, S)$$

for any diagram $T \in \mathbb{K}(\Omega)$, and any $v \in V(T)$ such that Ω_v^T is a cylinder with n interior marked points, with $m + 2$ boundary marked points and also any boundary contour contained marked points.

Define the action $\rho(\Omega) : \Sigma(\Omega) \rightarrow \text{Aut}(H^*(\Omega))$ by $\sigma(l)(\Gamma, \sigma') = l(\Gamma, \sigma'\sigma)$.

Extend our definitions of $\mathbb{K}(\Omega)$, $H^*(\Omega)$ and $\rho(\Omega)$ on disconnected sums $\Omega = \Omega_1 \amalg \Omega_2$ by $\mathbb{K}(\Omega) = \mathbb{K}(\Omega_1) \otimes \mathbb{K}(\Omega_2)$, $H^*(\Omega) = H^*(\Omega_1)^* \otimes H^*(\Omega_2)^*$. The actions $\rho(\Omega_1)$ and $\rho(\Omega_2)$ generate the action $\rho(\Omega)$. This gives a possible to define $H^*(\Omega)$ and $\rho(\Omega)$ for any disconnected sums of oriented sphere and disks.

Let γ be a simple cut on Ω and $\Omega_\# = \Omega/\gamma$. Denote by $\mathbb{K}_\gamma(\Omega) \subset \mathbb{K}(\Omega)$ the subspace generated by all diagrams containing the cut γ . A diagram $T \in \mathbb{K}(\Omega_\#)$ generates a diagram $\tilde{T} \in \mathbb{K}_\gamma(\Omega)$. This correspondence defines a monomorphism $\varepsilon_\gamma : \mathbb{K}(\Omega_\#) \rightarrow \mathbb{K}_\gamma(\Omega) \subset K(\Omega)$ and thus a homomorphism $\varepsilon_\gamma^* : H^*(\Omega) \rightarrow H^*(\Omega_\#)$.

4.3 Now we introduce a stabilising functor \mathcal{F} on the category $\mathcal{C}_{0,2,1}^0$ with structure functor, generated by arbitrary data $\{A, x \mapsto x^*, B, y \mapsto y^*\}$.

For local orientations \mathcal{O} , generated by a global orientation we put $\mathcal{F}(\Omega, \mathcal{O}) = (H^*(\Omega), \rho(\Omega), \Sigma(\Omega))$.

For isomorphism $\phi : (\Omega, \mathcal{O}) \rightarrow (\Omega', \mathcal{O}')$ denote by $\mathcal{F}(\phi) : H^*(\Omega, \mathcal{O}) \rightarrow H^*(\Omega', \mathcal{O}')$, the isomorphism that it generates.

Change of global orientation $\psi : (\Omega, \mathcal{O}) \rightarrow (\Omega, \mathcal{O}')$ is generated by an isomorphism $\phi : (\Omega, \mathcal{O}) \rightarrow (\Omega', \mathcal{O}')$. Put $\mathcal{F}(\psi) = \mathcal{F}(\phi)$.

For cutting morphism $\eta : (\Omega, \mathcal{O}) \rightarrow (\Omega_\#, \mathcal{O}_\#)$ by a cut $\gamma \subset \Omega$ we consider that $\mathcal{F}(\eta)$ is generated by ε_γ^* .

By definition, topological field theory

$$\mathcal{N} = \{A, x \mapsto x^*, B, y \mapsto y^*, \mathcal{F}, \Phi_{(\Omega, \mathcal{O})}\}$$

with values in a stabilising functor \mathcal{F} is called a *stable field theory*.

Remark 4.1. It follows from Manin [9] that the codiagram space $H^*(\Omega)$ of a sphere with n special points coincides with the cohomological algebra $H^*(\bar{M}_{0,n}, \mathbb{K})$ of the Deligne–Mumford compactification of moduli space of spheres with n punctures. Thus, for $x^* = x$

stable field theory on 2D category $\mathcal{C}_{0,0,1}$ coincides with the Kontsevich–Manin cohomological field theory [7].

5. System of disk correlation functions

5.1 Let A and B be vector spaces with bases $\{\alpha_1, \dots, \alpha_n\} \subset A, \{\beta_1, \dots, \beta_m\} \subset B$ and involutions $x \mapsto x^*, y \mapsto y^* (x, x^* \in A, y, y^* \in B)$. Consider a collection of tensors $\{f_{r,\ell} : A^{\otimes r} \otimes B^{\otimes \ell} \rightarrow \mathbb{K}\}$. By $\langle x_1, \dots, x_k, y_1, \dots, y_\ell \rangle$ denote $f_{k,\ell}(x_1 \otimes \dots \otimes x_k \otimes y_1 \otimes \dots \otimes y_\ell)$, where $x_i \in A, y_i \in B$. We say that this collection is a *system of disk correlation function*, if the following conditions hold.

Axiom 5.1. The value $\langle x_1, \dots, x_k, y_1, \dots, y_\ell \rangle$ is invariant under any permutation of x_i and cyclic permutation of y_i .

Axiom 5.2. $\langle x_1, \dots, x_k, y_1, \dots, y_\ell \rangle = \langle x_1^*, \dots, x_k^*, y_\ell^*, y_{\ell-1}^*, \dots, y_1^* \rangle$.

Axiom 5.3. The bilinear forms $\langle x_i, x_j \rangle$ and $\langle y_i, y_j \rangle$ are nondegenerate. Denote by $F^{\alpha_i \alpha_j}$ and $F^{\beta_i \beta_j}$ the inverse matrices for $F_{\alpha_i \alpha_j} = \langle \alpha_i, \alpha_j \rangle$ and $F_{\beta_i \beta_j} = \langle \beta_i, \beta_j \rangle$.

Axiom 5.4. Let $x_1, x_2, x_3, x_4, x^i \in A$ and $\mathcal{A} = \{x^1, \dots, x^r\}$. Denote by $\langle \mathcal{A} | x_1, x_2 | x_3, x_4 \rangle$ the sum of all numbers as $\langle x_1, x_2, x^{\sigma(1)}, \dots, x^{\sigma(s)}, \alpha_i \rangle F^{\alpha_i \alpha_j} \langle \alpha_j, x_3, x_4, x^{\sigma(s+1)}, \dots, x^{\sigma(r)} \rangle$, where $0 \leq s \leq r, 1 \leq i, j \leq n$ and σ passes through all permutations of r indices.

The axiom says that

$$\langle \mathcal{A} | x_1, x_2 | x_3, x_4 \rangle = \langle \mathcal{A} | x_4, x_1 | x_2, x_3 \rangle$$

for any \mathcal{A} and x_1, x_2, x_3, x_4 .

Axiom 5.5. Let $x^i \in A, \mathcal{A} = \{x^1, \dots, x^r\}, y_1, y_2, y_3, y_4, y^i \in B$, and $\mathcal{B} = \{y^1, \dots, y^p\}$. Denote by $\langle \mathcal{A}, \mathcal{B} | y_1, y_2 | y_3, y_4 \rangle$ the sum of all numbers as $\langle x^{\sigma(1)}, \dots, x^{\sigma(s)}, y^{\xi(1)}, \dots, y^{\xi(p_1)}, y_1, y^{\xi(p_1+1)}, \dots, y^{\xi(p_2)}, y_2, y^{\xi(p_2+1)}, \dots, y^{\xi(p_3)}, \beta_i \rangle F^{\beta_i \beta_j} \langle \beta_j, x^{\sigma(s+1)}, \dots, x^{\sigma(r)}, y^{\xi(p_3+1)}, \dots, y^{\xi(p_4)}, y_3, y^{\xi(p_4+1)}, \dots, y^{\xi(p_5)}, y_4, y^{\xi(p_5+1)}, \dots, y^{\xi(p)} \rangle$, where $0 \leq s \leq r, 0 \leq p_1 \leq p_2 \leq p_3 \leq p_4 \leq p_5 \leq p, 1 \leq i, j \leq m, \sigma$ passes through all permutations of r indices and ξ passes through all permutations of p indices.

The axiom says that

$$\langle \mathcal{A}, \mathcal{B} | y_1, y_2 | y_3, y_4 \rangle = \langle \mathcal{A}, \mathcal{B} | y_4, y_1 | y_2, y_3 \rangle$$

for any \mathcal{A}, \mathcal{B} , and y_1, y_2, y_3, y_4 .

Axiom 5.6. Let $x, x^i \in A, \mathcal{A} = \{x^1, \dots, x^r\}, y_1, y_2, y^i \in B$, and $\mathcal{B} = \{y^1, \dots, y^p\}$. Denote by $\langle \mathcal{A}, \mathcal{B} | x | y_1, y_2 \rangle$ the sum of all numbers as $\langle x, x^{\sigma(1)}, \dots, x^{\sigma(s)}, y^{\xi(1)}, \dots, y^{\xi(p_1)}, \beta_i \rangle F^{\beta_i \beta_j} \langle \beta_j, x^{\sigma(s+1)}, \dots, x^{\sigma(r)}, y^{\xi(p_1+1)}, \dots, y^{\xi(p_2)}, y_1, y^{\xi(p_2+1)}, \dots, y^{\xi(p_3)}, y_2, y^{\xi(p_3+1)}, \dots, y^{\xi(p)} \rangle$, where $0 \leq s \leq r, 0 \leq p_1 \leq p_2 \leq p_3 \leq p, 1 \leq i, j \leq m, \sigma$ passes through all permutations of r indices and ξ passes through all permutations of p indices.

The axiom says that

$$\langle \mathcal{A}, \mathcal{B} | x | y_1, y_2 \rangle = \langle \mathcal{A}, \mathcal{B} | x | y_2, y_1 \rangle$$

for any $\mathcal{A}, \mathcal{B}, x$, and y_1, y_2 .

Axiom 5.7. Let $x_1, x_2, x^i \in A, \mathcal{A} = \{x^1, \dots, x^r\}, y, y^i \in B$, and $\mathcal{B} = \{y^1, \dots, y^p\}$. Denote by $\langle \mathcal{A}, \mathcal{B} | x_1, x_2 | y \rangle$ the sum of all numbers as $\langle x_1, x_2, x^{\sigma(1)}, \dots, x^{\sigma(s)}, \alpha_i \rangle F^{\alpha_i \alpha_j} \langle \alpha_j, x^{\sigma(s+1)}, \dots, x^{\sigma(r)}, y, y^{\xi(1)}, \dots, y^{\xi(p)} \rangle$, where $0 \leq s \leq r, 1 \leq i, j \leq n, \sigma$ passes through all permutations of r indices and ξ passes through all permutations of p indices.

Denote by $\langle \mathcal{A}, \mathcal{B} | x_1, y | x_2 \rangle \langle \mathcal{A}, \mathcal{B} | x_2, y | x_1 \rangle$ the sum of all numbers as $\langle x, x^{\sigma(1)}, \dots, x^{\sigma(s)}, y, y^{\xi(1)}, \dots, y^{\xi(p_1)}, \beta_i^1, y^{\xi(p_2+1)}, \dots, y^{\xi(p_3)}, \beta_i^2, y^{\xi(p_4+1)}, \dots, y^{\xi(p)} \rangle F^{\beta_i^1 \beta_j^1} F^{\beta_i^2 \beta_j^2} \langle \beta_j^1, x^{\sigma(s+1)}, \dots, x^{\sigma(t)}, y^{\xi(p_1+1)}, \dots, y^{\xi(p_2)} \rangle \langle \beta_j^2, x^{\sigma(t+1)}, \dots, x^{\sigma(r)}, y^{\xi(p_3+1)}, \dots, y^{\xi(p_4)} \rangle$, where $0 \leq s \leq t \leq r, 0 \leq p_1 \leq p_2 \leq p_3 \leq p_4 \leq p, 1 \leq i^1, j^1, i^2, j^2 \leq m, \sigma$ passes through all permutations of r indices and ξ passes through all permutations of p indices.

The axiom says that

$$\langle \mathcal{A}, \mathcal{B} | x_1, x_2 | y \rangle = \langle \mathcal{A}, \mathcal{B} | x_1, y | x_2 \rangle \langle \mathcal{A}, \mathcal{B} | x_2, y | x_1 \rangle$$

for any $\mathcal{A}, \mathcal{B}, x_1, x_2$, and y .

We say that a system of disk correlation function is *extended* if the following condition holds.

Axiom 5.8. Let $x^i \in A, \mathcal{A} = \{x^1, \dots, x^r\}, y_1, y_2^i \in B$, and $\mathcal{B} = \{y^1, \dots, y^p\}$. Denote by $\langle \mathcal{A}, \mathcal{B} | y_1, y_2 \rangle_a$ the sum of all numbers as $\langle x^{\sigma(1)}, \dots, x^{\sigma(s)}, y_1, y^{\xi(1)}, \dots, y^{\xi(q)}, \alpha_i \rangle F^{\alpha_i \alpha_j} \langle \alpha_j, x^{\sigma(s+1)}, \dots, x^{\sigma(r)}, y_2, y^{\xi(q+1)}, \dots, y^{\xi(p)} \rangle$, where $0 \leq s \leq r, 0 \leq q \leq p, 1 \leq i, j \leq n, \sigma$ passes through all permutations of r indices and ξ passes through all permutations of p indices.

Denote by $\langle \mathcal{A}, \mathcal{B} | y_1, y_2 \rangle_b$ the sum of all numbers as $\langle x^{\sigma(1)}, \dots, x^{\sigma(r)}, y_1, y^{\xi(1)}, \dots, y^{\xi(q_1)}, \beta_i, y^{\xi(q_1+1)}, \dots, y^{\xi(q_2)}, y_2, y^{\xi(q_2+1)}, \dots, y^{\xi(q_3)} \beta_j, y^{\xi(q_3+1)}, \dots, y^{\xi(p)} \rangle F^{\beta_i \beta_j}$, where $0 \leq q_1 \leq q_2 \leq q_3 \leq p, 1 \leq i, j \leq n, \sigma$ passes through all permutations of r indices and ξ passes through all permutations of p indices.

The axiom says that

$$\langle \mathcal{A}, \mathcal{B} | y_1, y_2 \rangle_a = \langle \mathcal{A}, \mathcal{B} | y_1, y_2 \rangle_b$$

for any $\mathcal{A}, \mathcal{B}, y_1$ and y_2 .

It is easy to prove that the axioms are fulfilled for any bases $\{\alpha_1, \dots, \alpha_n\} \subset A, \{\beta_1, \dots, \beta_m\} \subset B$ if they are fulfilled for one pair of such bases.

5.3 Let $\mathcal{N} = \{A, x \mapsto x^*, B, y \mapsto y^*, \mathcal{F}, \Phi(\Omega, \mathcal{O})\}$ be a stable field theory on the category $\mathcal{C}_{0,1,1}^0$. Consider the collection of tensors $\langle x_1, \dots, x_k, y_1, \dots, y_\ell \rangle_{\mathcal{N}} = \Phi(\Omega, \mathcal{O})(z)(\emptyset, 1)$. Here Ω is a sphere (if $\ell = 0$) or a disk (if $\ell > 0$) with k special interior points a_1, \dots, a_k and with ℓ special boundaries points $b_1, \dots, b_\ell, \emptyset$ is the empty diagram of Ω and 1 is identical permutation. We assume that the points b_1, \dots, b_ℓ are ordered by the orientation of disk Ω and $z = x_1 \otimes \dots \otimes x_k \otimes y_1 \otimes \dots \otimes y_\ell$, where x_i corresponds to a_i and y_i corresponds to b_i . We say that the tensors $\langle x_1, \dots, y_\ell \rangle_{\mathcal{N}}$ are *generated* by \mathcal{N} .

Theorem 5.1. *The correspondence $\mathcal{N} \mapsto \{(x_1, \dots, y_\ell)_\mathcal{N}\}$ is one-to-one correspondence between stable field theories on $\mathcal{C}_{0,1,1}$ and systems of disk correlation functions. If \mathcal{N} is stable field theories on $\mathcal{C}_{0,2,1}$ then $\{(x_1, \dots, y_\ell)_\mathcal{N}\}$ is an extended systems of disk correlation functions.*

Proof.

- (1) Prove that $\{(x_1, \dots, y_\ell)_\mathcal{N}\}$ is a (extended) system of disk correlation functions. Axioms (5.1)–(5.3) for systems of disk correlation functions follow from Axioms (1)–(3) for topological field theories. Axioms 5.4–5.7 are fulfilled because $\Phi_{(\Omega, \mathcal{O})}(z) \in H^*(\Omega)$.
- (2) Now let $\{(x_1, \dots, y_\ell)\}$ be a system of disk correlation functions. Consider a sphere (if $\ell = 0$) or a disk (if $\ell > 0$) Ω with k special interior points a_1, \dots, a_k and with ℓ special boundaries points b_1, \dots, b_ℓ . Put $\Phi_{(\Omega, \mathcal{O})}(z)(\emptyset, 1) = (x_1, \dots, x_k, y_1, \dots, y_\ell)_\mathcal{N}$. We assume that the points b_1, \dots, b_ℓ are ordered by the orientation of disk Ω and $z = x_1 \otimes \dots \otimes x_k \otimes y_1 \otimes \dots \otimes y_\ell$, where x_i corresponds to a_i and y_i corresponds to b_i . \emptyset is the empty diagram of Ω and 1 is identical permutation.

Using Axioms (0), (4) and (5) for topological field theories we can uniquely continue $\Phi_{(\Omega, \mathcal{O})}(z)$ up to a linear function $l : \mathbb{K}(\Omega, \mathcal{O}) \rightarrow \mathbb{K}$. It follows from Axioms 5.4–5.7 for systems of disk correlation functions that $l \in H^*(\Omega, \mathcal{O})$. It follows from Axioms 5.1–5.3 for systems of disk correlation functions that (1)–(3) for topological field theories are fulfilled. □

6. Structure equations

6.1 Let, as above, $A, x \mapsto x^*$ and $B, y \mapsto y^*$ be vector spaces with involutions. These involutions generate involutions $*$ on $A^{\otimes k} \otimes B^{\otimes \ell}$ by the rule $*(x_1 \otimes \dots \otimes x_k \otimes y_1 \otimes \dots \otimes y_\ell) = (x_1^* \otimes \dots \otimes x_k^* \otimes y_\ell^* \otimes \dots \otimes y_1^*)$. Fix bases $\{\alpha_1, \dots, \alpha_n\} \subset A$ and $\{\beta_1, \dots, \beta_m\} \subset B$. By definition, we consider that a multiplication of monomials $\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell}$ and $\alpha_{\tilde{i}_1} \otimes \dots \otimes \alpha_{\tilde{i}_k} \otimes \beta_{\tilde{j}_1} \otimes \dots \otimes \beta_{\tilde{j}_\ell}$ is the monomial $\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} \otimes \alpha_{\tilde{i}_1} \otimes \dots \otimes \alpha_{\tilde{i}_k} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell} \otimes \beta_{\tilde{j}_1} \otimes \dots \otimes \beta_{\tilde{j}_\ell}$.

Now consider a tensor series, i.e., a formal series of tensor monomials $F = \sum c(i_1, \dots, i_k | j_1, \dots, j_\ell) \alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell}$, where $c(i_1, \dots, i_k | j_1, \dots, j_\ell) \in \mathbb{K}$. The involution $*$ naturally acts on the set of such series. The multiplication of monomial defines the multiplication of formal series.

Define “partial derivative” of tensor series. It is a linear continuation of partial derivative of monomials.

Let $\partial(\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell}) / \partial \alpha_i$ be the sum of monomials as $\alpha_{i_1} \otimes \dots \otimes \alpha_{i_{p-1}} \otimes \alpha_{i_{p+1}} \otimes \dots \otimes \alpha_{i_k} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell}$ such that $i_p = i$.

Similarly, let $\partial(\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell}) / \partial \beta_i$ be the sum of monomials as $\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_{p-1}} \otimes \beta_{j_{p+1}} \otimes \dots \otimes \beta_{j_\ell}$ such that $j_p = i$.

Put $\partial^2 / \partial \alpha_i \partial \alpha_j = (\partial / \partial \alpha_i)(\partial / \partial \alpha_j)$, $\partial^2 / \partial \alpha_i \partial \beta_j = (\partial / \partial \alpha_i)(\partial / \partial \beta_j)$, $\partial^2 / \partial \beta_i \partial \beta_j = (\partial / \partial \beta_i)(\partial / \partial \beta_j)$, $\partial^3 / \partial \alpha_i \partial \alpha_j \partial \alpha_r = (\partial / \partial \alpha_i)(\partial / \partial \alpha_j)(\partial / \partial \alpha_r)$.

A definition of $\partial^3(\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell}) / \partial\beta_i \partial\beta_j \partial\beta_r$ is more complicated. It is the sum of monomials as $\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} \otimes \beta_{s_2} \otimes \dots \otimes \beta_{s_{p-1}} \otimes \beta_{s_{p+1}} \otimes \dots \otimes \beta_{s_{q-1}} \otimes \beta_{s_{q+1}} \otimes \beta_{s_\ell}$ such that the sequences $\beta_i, \beta_{s_2}, \dots, \beta_{s_{p-1}}, \beta_j, \beta_{s_{p+1}}, \dots, \beta_{s_{q-1}}, \beta_r, \beta_{s_{q+1}}, \dots, \beta_{s_\ell}$ and $(\beta_{j_1}, \dots, \beta_{j_\ell})$ coincide other cyclic permutation.

Monomials $\alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell}$ and $\alpha_{\tilde{i}_1} \otimes \dots \otimes \alpha_{\tilde{i}_k} \otimes \beta_{\tilde{j}_1} \otimes \dots \otimes \beta_{\tilde{j}_\ell}$ are called *equivalent*, if $\bigcup_{r=1}^k i_r = \bigcup_{r=1}^k \tilde{i}_r$ and $\bigcup_{r=1}^\ell j_r = \bigcup_{r=1}^\ell \tilde{j}_r$. By $[\alpha_{i_1} \otimes \dots \otimes \beta_{j_\ell}]$ denote the equivalence class of $\alpha_{i_1} \otimes \dots \otimes \beta_{j_\ell}$. A tensor series $F = \sum c(i_1, \dots, i_k | j_1, \dots, j_\ell) a_{i_1} \otimes \dots \otimes b_{j_\ell}$ generates a series $[F] = \sum c[i_1, \dots, i_k | j_1, \dots, j_\ell] [\alpha_{i_1} \otimes \dots \otimes \beta_{j_\ell}]$, where the sum is taken over equivalence classes of monomials and $c[i_1, \dots, i_k | j_1, \dots, j_\ell]$ is the sum of all coefficients $c(\tilde{i}_1, \dots, \tilde{j}_\ell)$, corresponding to monomials from the equivalence class $[\alpha_{i_1} \otimes \dots \otimes \beta_{j_\ell}]$.

We say that a tensor series $F = \sum c(i_1, \dots, i_k | j_1, \dots, j_\ell) \alpha_{i_1} \otimes \dots \otimes \alpha_k \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell}$ is a *structure series* on a space $H = A \oplus B$ with an involution $*$ and bases $\{\alpha_1, \dots, \alpha_n | \beta_1, \dots, \beta_m\}$, if the following conditions hold.

Axiom 6.1. The coefficients $c(i_1, \dots, j_k | j_1, \dots, j_\ell)$ are invariant under all permutations of $\{i_r\}$ and cyclic permutations of $\{j_r\}$.

Axiom 6.2. $F^* = F$.

Axiom 6.3. The coefficients $c(i, j)$ and $c(|i, j)$ generate nondegenerate matrices. By $F_a^{\alpha_i \alpha_j}$ and $F_b^{\beta_i \beta_j}$ denote the inverse matrices of $c(i, j)$ and $c(|i, j)$, respectively.

Axiom 6.4.

$$\left[\sum_{p,q=1}^n \frac{\partial^3 F_a}{\partial \alpha_i \partial \alpha_j \partial \alpha_p} \otimes F_a^{\alpha_p \alpha_q} \frac{\partial^3 F_a}{\partial \alpha_q \partial \alpha_k \partial \alpha_\ell} \right] = \left[\sum_{p,q=1}^n \frac{\partial^3 F_a}{\partial \alpha_k \partial \alpha_j \partial \alpha_p} \otimes F_a^{\alpha_p \alpha_q} \frac{\partial^3 F_a}{\partial \alpha_q \partial \alpha_i \partial \alpha_\ell} \right],$$

where F_a is the part of the tensor series F , consisting from all monomials without monomials, containing some β_i .

Axiom 6.5.

$$\left[\sum_{p,q=1}^m \frac{\partial^3 F}{\partial \beta_i \partial \beta_j \partial \beta_p} \otimes F_b^{\beta_p \beta_q} \frac{\partial^3 F}{\partial \beta_q \partial \beta_k \partial \beta_\ell} \right] = \sum_{p,q=1}^m \left[\frac{\partial^3 F}{\partial \beta_\ell \partial \beta_i \partial \beta_p} \otimes F_b^{\beta_p \beta_q} \frac{\partial^3 F}{\partial \beta_q \partial \beta_j \partial \beta_k} \right].$$

Axiom 6.6.

$$\left[\sum \frac{\partial^2 F}{\partial \alpha_k \partial \beta_p} \otimes F_b^{\beta_p \beta_q} \frac{\partial^3 F}{\partial \beta_q \partial \beta_i \partial \beta_j} \right] = \left[\sum \frac{\partial^2 F}{\partial \alpha_k \partial \beta_p} \otimes F_b^{\beta_p \beta_q} \frac{\partial^3 F}{\partial \beta_q \partial \beta_j \partial \beta_i} \right].$$

Axiom 6.7.

$$\left[\sum \frac{\partial^2 F}{\partial \beta_k \partial \alpha_p} \otimes F_a^{\alpha_p \alpha_q} \frac{\partial^3 F}{\partial \alpha_q \partial \alpha_i \partial \alpha_j} \right] = \left[\sum \frac{\partial^2 F}{\partial \alpha_i \partial \beta_p} \otimes F_b^{\beta_p \beta_q} \frac{\partial^3 F}{\partial \beta_q \partial \beta_k \partial \beta_r} \otimes F_b^{\beta_r \beta_\ell} \frac{\partial^2 F}{\partial \beta_\ell \partial \alpha_j} \right].$$

We say that the conditions from **Axioms 6.1–6.7** are *structure equations*. They are noncommutative analogues of associativity equations [5,14].

6.2 Any noncommutative tensor series $F = \sum c(i_1, \dots, i_k | j_1, \dots, j_\ell) \alpha_{i_1} \otimes \dots \otimes \alpha_{i_k} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell}$ generates a family of tensors $f_{r,\ell}^F : A^{\otimes r} \otimes B^{\otimes \ell} \rightarrow \mathbb{C}$, where $f_{r,\ell}^F(\alpha_{i_1} \otimes \dots \otimes \alpha_{i_r} \otimes \beta_{j_1} \otimes \dots \otimes \beta_{j_\ell}) = c(i_1, \dots, i_r | j_1, \dots, j_\ell)$. Put $\langle x_{i_1}, \dots, x_{i_r}, y_{j_1}, \dots, y_{j_\ell} \rangle_F = f_{r,\ell}^F(x_{i_1} \otimes \dots \otimes x_{i_r} \otimes y_{j_1} \otimes \dots \otimes y_{j_\ell})$.

Theorem 6.1. *A tensor series F is a structure series, iff the collection of tensors $\{\langle x_1, \dots, x_r, y_1, \dots, y_\ell \rangle_F\}$ forms a system of disk correlation functions. Any system of disk correlation function is generated by a structure series.*

Proof. Formal calculation demonstrates equivalence of the axioms of disk correlation functions and the axioms of structure series with the same numbers. □

7. Noncommutative Frobenius manifolds

7.1 According to [6], any solution WDVV equation generates a special deformation of Frobenius algebra. In this section we associate to any solution of structure equation some deformation of extended Frobenius algebra.

Definition 7.1. Extended Frobenius algebra $\mathcal{H} = \{H = A + B, (x, y), x \mapsto x^*\}$ is a finite dimensional associative algebra H over \mathbb{K} endowed with a decomposition $H = A + B$ of H into a direct sum of vector spaces; an invariant symmetric bilinear form $(x, y) : H \otimes H \rightarrow \mathbb{K}$ that is $(x, y) = (y, x)$ and $(xy, z) = (x, yz)$; an involutive anti-automorphism $H \rightarrow H$, denoted by $x \mapsto x^*$; such that the following axioms hold:

- (1) A is a subalgebra belonging to the centre of algebra H ;
- (2) B is a two-sided ideal of H (typically noncommutative);
- (3) restrictions $(x, y)|_A$ and $(x, y)|_B$ are nondegenerate scalar products on algebras A and B , resp.
- (4) an involutive anti-automorphism preserves the decomposition $H = A + B$ and the form (x, y) on H , i.e. $A^* = A, B^* = B, (x^*, y^*) = (x, y)$.

A full description of semisimple extended Frobenius algebras over \mathbb{C} follows from [3, Section 2].

The structure tensors $F_{\alpha',\alpha''} = (\alpha', \alpha''), F_{\beta',\beta''} = (\beta', \beta''), R_{\alpha\beta} = (\alpha, \beta), S_{\alpha',\alpha'',\alpha'''} = (\alpha', \alpha'', \alpha'''), T_{\beta',\beta'',\beta'''} = (\beta', \beta'', \beta'''), R_{\alpha\beta'\beta''} = (\alpha\beta', \beta''), I_{\alpha',\alpha''} = ((\alpha')^*, \alpha''), I_{\beta',\beta''} = ((\beta')^*, \beta'')$, uniquely describe an extended Frobenius algebra [3]. Here we denote by $\alpha, \alpha', \alpha''$ and β, β', β'' the elements of the bases $\{\alpha_1, \dots, \alpha_n\} \subset A$ and $\{\beta_1, \dots, \beta_m\} \subset B$, respectively.

7.2 Later we assume that the field \mathbb{K} is the field of real or complex numbers. Associate a deformation of extended Frobenius algebras to a structure series on a space $H = A + B$ with involution $* : H \rightarrow H$ and a basis $\{\alpha_1, \dots, \beta_\ell\} \subset H$. First consider the coordinates on H that associate the element $z = s^i \alpha_i + t^j \beta_j$ to collections $(s^1, \dots, s^n | t^1, \dots, t^m) \in \mathbb{K}^{n+m}$. The

correspondences $\alpha_i \mapsto s^i$, $\beta_j \mapsto t^j$ and (tensor multiplication) \mapsto (number multiplication) map tensor series $F(\alpha_1, \dots, \beta_m)$ to formal number series $\hat{F}(s^1, \dots, t^m)$. By definition, put $F_{\alpha_i \alpha_j} = c(i, j)$, $F_{\beta_i \beta_j} = c(j, i)$, $I_{\alpha_i \alpha_j} = F_{\alpha_i \alpha_j^*}$, $I_{\beta_i \beta_j} = F_{\beta_i \beta_j^*}$. Let $\{F^{\alpha_i \alpha_j}\}$ and $\{F^{\beta_i \beta_j}\}$ be the inverse matrix for $\{F_{\alpha_i \alpha_j}\}$ and $\{F_{\beta_i \beta_j}\}$, respectively. Put

$$R_{\alpha_i \beta_j}^F = \frac{\partial^2 F}{\partial \alpha_i \partial \beta_j}, \quad S_{\alpha_i \alpha_j \alpha_k}^F = \frac{\partial^3 F}{\partial \alpha_i \partial \alpha_j \partial \alpha_k}, \quad T_{\beta_i \beta_j \beta_k}^F = \frac{\partial^3 F}{\partial \beta_i \partial \beta_j \partial \beta_k},$$

$$R_{\alpha_i \beta_j \beta_k}^F = \frac{\partial^2 F}{\partial \alpha_i \partial \beta_p} F^{\beta_p \beta_q} \frac{\partial F}{\partial \beta_q \partial \beta_j \partial \beta_k}.$$

Theorem 7.1. *Let F be a structure series. Then at convergence points the tensors $F_{\alpha_i \alpha_j}$, $F_{\beta_i \beta_j}$, $I_{\alpha_i \alpha_j}$, $I_{\beta_i \beta_j}$, $\hat{R}_{\alpha_i \beta_j} = \hat{R}_{\alpha_i \beta_j}^F(s^1, \dots, t^m)$, $\hat{S}_{\alpha_i \alpha_j \alpha_k} = \hat{S}_{\alpha_i \alpha_j \alpha_k}^F(s^1, \dots, t^m)$, $\hat{T}_{\beta_i \beta_j \beta_k} = \hat{T}_{\beta_i \beta_j \beta_k}^F(s^1, \dots, t^m)$, $\hat{R}_{\alpha^i \beta^j \beta^k} = \hat{R}_{\alpha^i \beta^j \beta^k}^F(s^1, \dots, t^m)$ define an extended Frobenius algebra.*

Proof. We shall raise indices of tensors, using the tensors $F^{\alpha_i \alpha_j}$ and $F^{\beta_i \beta_j}$. In the case of nonsymmetric tensors we always raise the last index. By definition, put $\hat{S}_{\alpha_i \alpha_j \alpha_k \alpha_\ell} = \sum_r \hat{S}_{\alpha_i \alpha_j}^{\alpha_r} \hat{S}_{\alpha_r \alpha_k \alpha_\ell} \hat{T}_{\beta_i \beta_j \beta_k \beta_\ell} = \sum_r \hat{T}_{\beta_i \beta_j}^{\beta_r} \hat{T}_{\beta_r \beta_k \beta_\ell}$. It follows from [3] that the above theorem is equivalent to the following conditions:

- (1) matrices $F_{\alpha_i \alpha_j}$ and $F_{\beta_i \beta_j}$ are nondegenerate;
- (2) tensors $\hat{S}_{\alpha_i \alpha_j \alpha_k}$ and $\hat{S}_{\alpha_i \alpha_j \alpha_k \alpha_\ell}$ are symmetric with respect to all permutations;
- (3) tensors $\hat{T}_{\beta_i \beta_j \beta_k}$ and $\hat{T}_{\beta_i \beta_j \beta_k \beta_\ell}$ are symmetric with respect to cyclic permutations;
- (4) $\hat{R}_{\alpha, \beta_1, \beta_2} = \hat{R}_{\alpha}^{\beta'} \hat{T}_{\beta', \beta_1, \beta_2}$;
- (5) $\hat{R}_{\beta}^{\alpha'} \hat{S}_{\alpha', \alpha_1, \alpha_2} = \hat{R}_{\alpha_1}^{\beta'} \hat{R}_{\alpha_2}^{\beta''} \hat{T}_{\beta', \beta'', \beta}$;
- (6) $\hat{R}_{\alpha, \beta_1, \beta_2} = \hat{R}_{\alpha, \beta_2, \beta_1}$;
- (7) $\hat{I}_{\alpha_1}^{\alpha'} \hat{I}_{\alpha', \alpha_2} = F_{\alpha_1, \alpha_2}$, $\hat{I}_{\beta_1}^{\beta'} \hat{I}_{\beta', \beta_2} = F_{\beta_1, \beta_2}$;
- (8) $\hat{I}_{\alpha_1, \alpha_2} = \hat{I}_{\alpha_2, \alpha_1}$, $\hat{I}_{\beta_1, \beta_2} = \hat{I}_{\beta_2, \beta_1}$, $\hat{I}_{\alpha'}^{\alpha} \hat{R}_{\alpha', \beta} = \hat{I}_{\beta}^{\beta'} \hat{R}_{\alpha, \beta'}$;
- (9) $\hat{I}_{\alpha_1}^{\alpha'} \hat{I}_{\alpha_2}^{\alpha''} \hat{I}_{\alpha_3}^{\alpha'''} \hat{S}_{\alpha''', \alpha'', \alpha'} = \hat{S}_{\alpha_1, \alpha_2, \alpha_3}$, $\hat{I}_{\beta_1}^{\beta'} \hat{I}_{\beta_2}^{\beta''} \hat{I}_{\beta_3}^{\beta'''} \hat{T}_{\beta''', \beta'', \beta'} = \hat{T}_{\beta_1, \beta_2, \beta_3}$.

All these conditions directly follow from our definition and the axioms of structure series. □

Thus, structure series generate an extended Frobenius algebra in the points of H , where the series from Theorem 7.1 converge. This defines a deformation of extended Frobenius algebra, i.e. some analogue of Frobenius manifold [6] for (noncommutative) extended Frobenius algebras.

8. Examples

8.1 For $\dim B = 0$, Theorems 5.1 and 6.1 get over into Theorem III, Section 4.3 from Manin [9] that claim that cohomological field theory is equivalent to formal Frobenius manifolds.

Noncommutative Frobenius manifolds with $\dim A = 0$, $\dim B = 1$ are Frobenius manifolds of range 1. They are described by formal series $F(\beta)$ [9].

For $\dim A = 1$, $\dim B = 1$. Put $\langle \alpha^n, \beta^m \rangle = \langle \alpha, \dots, \alpha, \beta, \dots, \beta \rangle$. **Axiom 6.7** demonstrates that all correlators are determined by correlators $\langle \alpha^n \rangle$, $\langle \beta^m \rangle$ and $\langle \alpha, \beta \rangle$.

Suppose, for example that $\langle \alpha, \alpha \rangle = \langle \alpha, \alpha, \alpha \rangle = \langle \beta, \beta \rangle = \langle \beta, \beta, \beta \rangle = 1$ and $\langle \alpha^m \rangle = \langle \beta^m \rangle = 0$ for $m > 3$. Then according to **Axiom 6.7**, $\langle \alpha^3 \rangle \langle \alpha, \beta \rangle = \langle \beta^3 \rangle \langle \alpha, \beta \rangle \langle \alpha, \beta \rangle$ and thus $\langle \alpha, \beta \rangle$ is equal to 0 or 1. Moreover, **Axiom 6.7** gives $\langle \alpha^3 \rangle \langle \alpha \beta^2 \rangle = 2 \langle \alpha, \beta \rangle \langle \beta^4 \rangle \langle \alpha, \beta \rangle + 2 \langle \alpha, \beta \rangle \langle \beta^3 \rangle \langle \alpha, \beta^2 \rangle$ and thus $\langle \alpha, \beta^2 \rangle = 0$. Analogously we prove that $\langle \alpha, \beta^m \rangle = 0$ for $m > 1$. On the other hand, **Axiom 6.7** gives $\langle \alpha^3 \rangle \langle \alpha^2, \beta \rangle = \langle \alpha, \beta \rangle \langle \alpha \beta \rangle \langle \alpha, \beta^3 \rangle + 2 \langle \alpha^2, \beta \rangle \langle \alpha, \beta \rangle \langle \beta^3 \rangle$ and thus $\langle \alpha^2, \beta \rangle = 0$. Analogously we prove that $\langle \alpha^n, \beta^m \rangle = 0$ for $nm > 1$.

Thus, if the restriction of structure series F on A and B is $(1/2)\alpha^{\otimes 2} + \alpha^{\otimes 3}$ and $(1/2)\beta^{\otimes 2} + \beta^{\otimes 3}$ then $F = (1/2)a^{\otimes 2} + a^{\otimes 3} + (1/2)b^{\otimes 2} + b^{\otimes 3} + \xi \alpha \otimes \beta$, where $\xi \in \{0, 1\}$.

8.2 In analogy with [12] any extended Frobenius algebra $\mathcal{H} = \{H = A + B, (\cdot, \cdot), *\}$ with unite 1 generates some system of disk correlation functions by the rule $\langle \alpha^1, \dots, \alpha^s, \beta^1, \dots, \beta^t \rangle = \langle \alpha^1, \dots, \alpha^s \beta^1, \dots, \beta^t, 1 \rangle$.

8.3 According to [4], a noncommutative Frobenius manifold in our sense is generated by a topological Landau–Ginzburg theory of B -type with 2D-brane conditions.

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